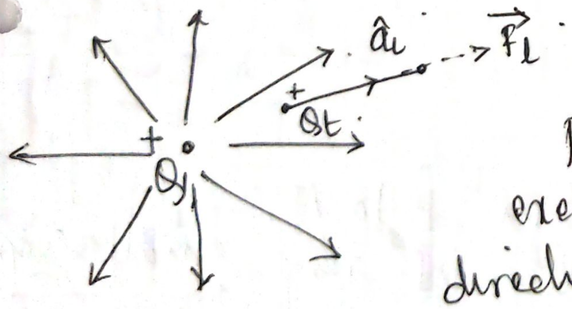
Energy And Potential

*** Energy expended in moving a point charge in an electric field

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field.

If we attempt to move the test charge against electric field, we have to exert a force equal & opposite to that exerted by the field, and this requires to expend energy to do work.

Consider a positive charge Q_1 and its electric field $\vec{E}$. If a positive test charge Q_t is placed in this field, it will move due to the force of repulsion. Let the movement of charge Q_t is dl. The direction in which the movement has taken place is denoted by unit vector $\hat{a}_L$, in the direction of dl.



According to coulomb's law

$$\vec{F} = Q_t\vec{E}.$$

But the component of this force exerted by the field in the direction dl, is responsible to move charge Q_t, through the distance dl.

$$\vec{F_L} = \vec{F} \cdot \bar{a}_L = Q_t\vec{E} \cdot \bar{a}_L \ N$$

To keep in equilibrium, it is necessary to apply the force which is equal & opposite to the force exerted by the field in the direction dl.

$$\vec{F}_{applied} = -\vec{F_L} = -Q_t\vec{E} \cdot \hat{a}_L.$$

differential work done by an external source in moving the charge Q_t through a distance dl, against the direction of field $\vec{E}$ is given by. ②

$$dW = \vec{F}_{applied} \times dl = -Q_t\vec{E} \cdot \hat{a}_l \cdot \vec{dl}$$

$$\vec{dl} = dl\,\hat{a}_l$$

$$\boxed{dW = -Q_t\vec{E} \cdot \vec{dl}} \quad J$$

$$\Rightarrow \boxed{W = -Q\int_{initial}^{final} \vec{E} \cdot \vec{dl}}$$

★ Line Integral

The work done in moving a charge from initial to final position is.

$$W = -Q\int_{B}^{A} \vec{E} \cdot \vec{dl}$$

where $\vec{E} \cdot \vec{dl}$ gives the component of $\vec{E}$ along the direction $\vec{dl}$.

Steps:

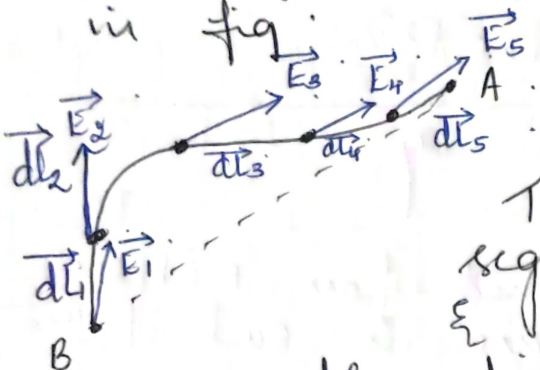
i) Choose any arbitrary path from B to A.

ii) Break up the path into number of very small segments which are called differential lengths.

iii) Find component of $\vec{E}$ along each segments.

iv) Adding all such components and multiplying by charge, the required work done can be obtained.

Consider an uniform electric field $\vec{E}$. The charge is moved from B to A along the path as shown in fig. ③



The path B to A is divided into number of small segments. The various distance vectors along the segments chosen are $\vec{dl_1}, \vec{dl_2}, \vec{dl_3}, \vec{dl_4}$ & $\vec{dl_5}$ while the electric field in these directions are $\vec{E_1}, \vec{E_2}, \vec{E_3}, \vec{E_4}$ & $\vec{E_5}$.

Hence the line integral from B to A can be expressed as

$$W = -Q\left(\vec{E_1}\cdot\vec{dl_1} + \vec{E_2}\cdot\vec{dl_2} + \vec{E_3}\cdot\vec{dl_3} + \vec{E_4}\cdot\vec{dl_4} + \vec{E_5}\cdot\vec{dl_5}\right]$$

But the electric field is uniform, so,

$$\vec{E_1} = \vec{E_2} = \vec{E_3} = \vec{E_4} = \vec{E_5}$$

$$W = -Q\vec{E}\cdot\left[\vec{dl_1} + \vec{dl_2} + \vec{dl_3} + \vec{dl_4} + \vec{dl_5}\right)$$

Now $\vec{dl_1} + \vec{dl_2} + \dots \vec{dl_5}$ is the vector addition. So according to method of polygon the sum of all such vectors is the vector joing initial to final point

$$\therefore W = -Q\vec{E}\cdot\vec{L_{BA}}$$

In general, $\boxed{W = -Q\int_B^A \vec{E}\cdot\vec{dl}}$

Thus the work done in moving a charge from one location B to another A, in a static, uniform or nonuniform electric field $\vec{E}$ is independent of the path selected. The line integral of $\vec{E}$ is determined completely by the endpoints B & A of the path & not the actual path selected. This is called conservative property of electric field $\vec{E}$ & field $\vec{E}$ is said to be conservative.

Cartesian $\overline{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$.
Cylindrical $\overline{dl} = dr\hat{a}_r + r\,d\phi\,\hat{a}_\phi + dz\hat{a}_z$.
Spherical $\overline{dl} = dr\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$;

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(1). An electrostatic field is given by.
$\overline{E} = -8xy\hat{a}_x - 4x^2\hat{a}_y + \hat{a}_z$ V/m.
The charge of 6C is to be moved from B(1, 8, 5) to A(2, 18, 6). Find the work done in each of the following cases.
1. The path selected is $y = 3x^2 + z$, $z = x + 4$.
2. The straight line from B to A.
Show that work done remains same & is independent of the path selected.

:- $$W = -Q\int_B^A \overline{E}\cdot\overline{dl}$$

$$\overline{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

$$\overline{E}\cdot\overline{dl} = (-8xy\hat{a}_x - 4x^2\hat{a}_y + \hat{a}_z)(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= -8xy\,dx - 4x^2\,dy + dz.$$

$$W = -Q\int_B^A [-8xy\,dx - 4x^2\,dy + dz]$$

$$W = -Q\left[\int_B^A -8xy\,dx - \int_B^A 4x^2\,dy + \int_B^A dz\right]$$

Case i) :- The path $y = 3x^2 + z$, $z = x + 4$.

$$dy = (6x + 1)\,dx.$$

(5)

$$W = -Q\left\{\int_{x=1}^{2} -8x\left[3x^2+x+4\right]dx - \int_{x=1}^{2} 4x^2(6x+1)\,dx + \int_{z=5}^{6} dz\right\}$$

$$= -Q\left\{\int_{x=1}^{2}\left[-24x^3-8x^2-32x\right]dx - \int_{x=1}^{2}(24x^3+4x^2)\,dx + \int_{z=5}^{6} dz\right\}$$

$$= -Q\left\{\left[-6x^4-\frac{8}{3}x^3-16x^2-6x^4-\frac{4}{3}x^3\right]_{x=1}^{2} + \left[z\right]_{z=5}^{6}\right\}$$

$$= -Q\left\{-256+1\right\} = 6\times 256 = \boxed{1530\ J}$$

<u>Case ii)</u> :-

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{z-z_1}{y-y_1} = \frac{z_2-z_1}{y_2-y_1}$$

$$\frac{x-x_1}{z-z_1} = \frac{x_2-x_1}{z_2-z_1}$$

using coordinates of A & B.

$$y-8 = \frac{18-8}{2-1}(x-1)$$

$$y-8 = 10(x-1)$$

$$\boxed{y = 10x-2}$$

$$\boxed{dy = 10dx}$$

& $z-5 = \dfrac{6-5}{18-8}(y-8) \Rightarrow 10z = y+42$

$$\therefore W = -Q\left[\int_{x=1}^{2} -8xy\,dx - \int_{y=8}^{18} 4x^2 dy + \int_{z=5}^{6} dz\right].$$

$$= -Q\left[\int_{x=1}^{2} -8x(10x-2)dx - \int_{x=1}^{2} 4x^2(10dx) + \int_{z=5}^{6} dz\right]$$

$$= -Q\left\{\left[\frac{-80}{3}x^3 + \frac{16x^2}{2} - \frac{40x^3}{3}\right]_{x=1}^{2} + [z]_5^6\right\}$$

$$= -Q\{-213.33 + 32 - 106.667 + 26.667 - 8 + 13.33 + 1\}$$

$$= Q[-255] = 1530\text{ J}$$

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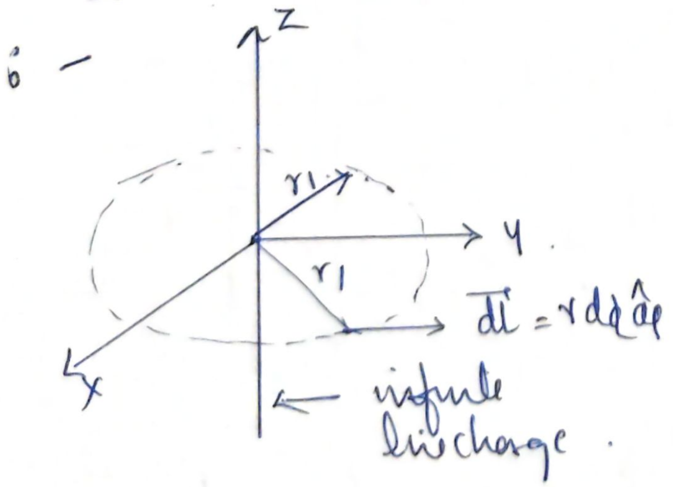
(2). Consider an infinite line charge along z-axis. Show that the work done is zero if a point charge Q is moving in a circular path of radius r_1, centred at the line charge



$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\hat{a}_r$$ (electric field due to infinite line charge)

$$\vec{dl} = r\,d\phi\,\hat{a}_\phi$$

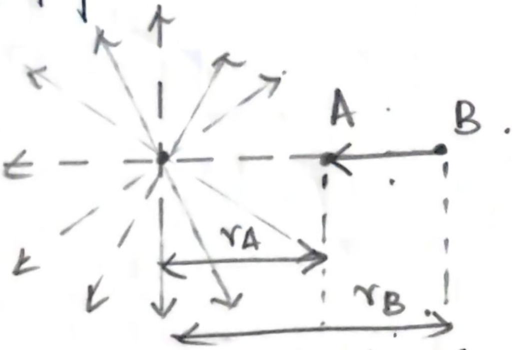
$$W = -Q\int_{\text{initial}}^{\text{final}} \vec{E}\cdot\vec{dl}$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0 r}\hat{a}_r \cdot r\,d\phi\,\hat{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} r\,d\phi\,(\hat{a}_r\cdot\hat{a}_\phi) = 0.$$

Potential due to point Charge

→ Consider point charge located at the origin of a spherical co-ordinate system producing $\vec{E}$ radially in all the directions as in fig.



→ Field $\vec{E}$ due to a point charge Q at a radial distance r from origin is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

→ Consider a unit charge which is placed at point B which is at a distance r_B from origin.

→ $\vec{dl} = dr\hat{a}_r + r d\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi$

→ Potential difference V_{AB} between points A & B is given by.

$$V_{AB} = -\int_B^A \vec{E}\cdot\vec{dl}$$

$$V_{AB} = -\int_{r_B}^{r_A}\left[\frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r\right]\cdot\left[dr\hat{a}_r + r d\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi\right]$$

$$= -\int_{r_B}^{r_A}\frac{Q}{4\pi\epsilon_0 r^2}dr$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_{r_B}^{r_A}\frac{1}{r^2}dr = \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A}$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]}$$

Concept of Absolute Potential

Consider potential difference V_{AB} due to movement of unit charge from B to A in a field of a point charge Q is given by

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Let the charge is moved from infinity to the point A. ie $r_B = \infty$. Hence $\frac{1}{r_B} = \frac{1}{\infty} = 0$.

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = \frac{Q}{4\pi\epsilon_0 r_A}\ V.$$

$\therefore$ Potential of point A is denoted as,

$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 r_A}}\ V$$

This is called absolute potential of point A.

IIIly, absolute potential of point B is

$$\boxed{V_B = \frac{Q}{4\pi\epsilon_0 r_B}}\ V$$

This is work done in moving unit charge from infinity at to point B.

$\therefore$ Hence, potential difference can be expressed as the difference between the absolute potentials of two points.

$$\boxed{V_{AB} = V_A - V_B}\ V$$

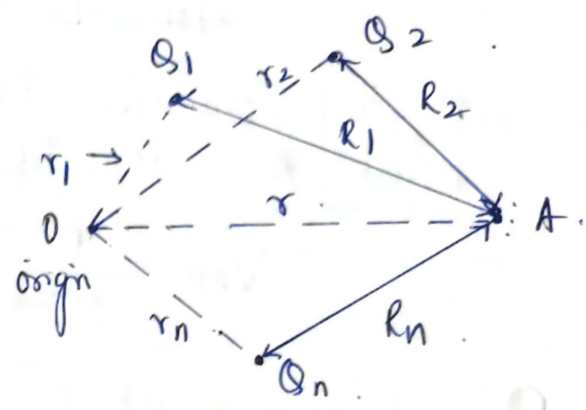
Thus absolute potential can be defined as,

Absolute potential at any point in an electric field is defined as that work done in moving a unit test charge from ∞ to the point, against the direction of the field.

∴ In general,

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$ when reference point is at infinity.

Potential due to several Point Charges.



Consider various point charges $Q_1, Q_2 \dots Q_n$ located at the distances $r_1, r_2 \dots r_n$ from the origin as shown in fig.

using superposition principle,

The potential V_{A1} ~~at~~ due to Q_1 is given by

$$V_{A1} = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} = \frac{Q_1}{4\pi\epsilon_0 R_1} \; V$$

where $R_1 = |r - r_1|$ = distance between point A and position of Q_1.

The potential V_{A2} due to Q_2 is given by

$$V_{A2} = \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\epsilon_0 R_2} \; V$$

potential ~~VAN~~ V_{AN} due to Q_N is given by

$$V_{An} = \frac{Q_n}{4\pi\epsilon_0 |r - r_n|} = \frac{Q_n}{4\pi\epsilon_0 R_n} \; V$$

As the potential is scalar, the net potential at point A is the algebraic sum of the potentials at A due to individual point charges, considered one at a time.

$$V_A = V_{A1} + V_{A2} + \cdots + V_{An}$$

$$= \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n}$$

$$\boxed{V_A = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |r - r_m|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 R_m} \; V}$$

Potential Calculation when reference is other than infinity

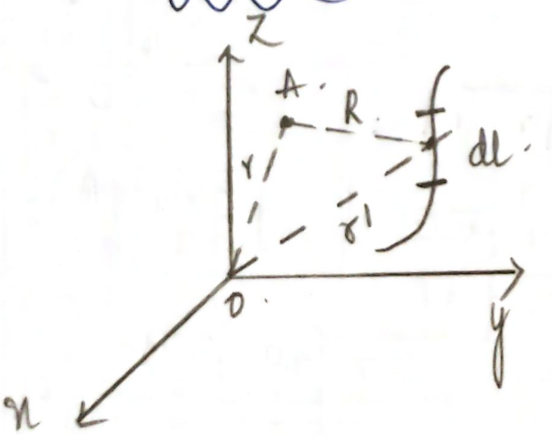
→ Uptill now we are under the assumption that the reference position of zero potential is at infinity.

→ If any other point than infinity is selected as the reference then the potential at a point A due to point charge Q at the origin becomes

$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 R_A} + C}$$

C = Constant to be determined at chosen reference point where V = 0.

Potential due to line charge



Consider line charge having density ρ_L C/m as in fig.

Consider differential length dL at a distance r' from the origin.

differential charge,

$$dQ = \rho_L \, dL$$

Let the potential at A due to differential length is

$$dV_A = \frac{dQ}{4\pi\epsilon_0 |r - r'|} = \frac{dQ}{4\pi\epsilon_0 R}$$

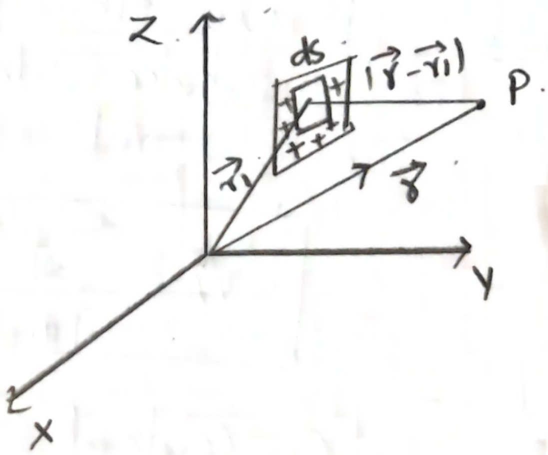
$R = |r - r'|$ indicates the distance of point A from the differential charge.

Total potential V_A due to entire length is given by

$$V_A = \int_{line} \frac{dQ}{4\pi\epsilon_0 R}$$

$$\boxed{V_A = \int_{line} \frac{\rho_L dL}{4\pi\epsilon_0 R}} \quad V$$

Potential due to Surface Charge.



Consider uniform surface charge density ρ_s C/m² on a surface as in fig.

Consider a differential surface ds at a distance r' from the origin,

differential charge

$$dQ = \rho_s ds.$$

Let the potential at point A due to differential surface is

$$dV_A = \frac{dQ}{4\pi\epsilon_0 |r - r'|} = \frac{\rho_s ds}{4\pi\epsilon_0 |r - r'|}.$$

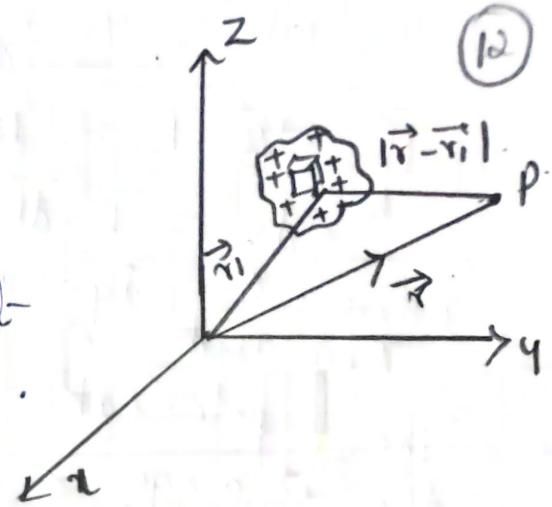
$R = |r - r'|$ indicates the distance of point A from the differential surface.

Total potential V_A due to entire surface is

$$V_A = \int_{surface} \frac{dQ}{4\pi\epsilon_0 R} \Rightarrow \boxed{V_A = \int_s \frac{\rho_s ds}{4\pi\epsilon_0 |r - r'|}}.$$

Volume charge Distribution :-

Consider a uniform volume charge density ρ_v C/m³ over the given volume as in fig.



Consider a differential volume dv at a distance r' from the origin.

differential charge

$$dq = \rho_v \, dv$$

Let the potential at A due to differential volume is

$$dV_A = \frac{dq}{4\pi\epsilon_0 |r-r'|} = \frac{dq}{4\pi\epsilon_0 R}.$$

$R = |r-r'|$ = distance of point A from the differential charge.

Total potential V_A due to entire volume is

$$V_A = \int_V \frac{dq}{4\pi\epsilon_0 R}$$

$$\boxed{V_A = \int_V \frac{\rho_v \, dv}{4\pi\epsilon_0 R}}$$

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* Relation between Electric field intensity and electric potential

→ Consider two neighbouring points A (x, y, z) and B(x+dx, y+dy, z+dz) separated by a small distance $\vec{dl}$ in an electric field $\vec{E}$

→ Let the potential at B be higher than that at A by an amount dV.

∴ work done in moving a charge Q from A to B is $dw = -Q\vec{E}\cdot\vec{dl}$.

→ Potential is the work done against the field in moving a unit +ve charge from one point to the other $V = \frac{W}{Q} \Rightarrow dV = \frac{dW}{dQ}$.

If $dQ = 1$, then $dW = dV$.

$\therefore dV = -\vec{E} \cdot \vec{dl} = -\vec{E}(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$ —(1)

where $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are the unit vectors along x, y and z directions.

The potential difference dv can be expressed as the change in the potential V as we move from A to B, which is expressed as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz.$$

expressed as dot product.

$$dV = \left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right](dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z) \text{ —(2)}$$

from eq (1) & (2) we get

$$-\vec{E}(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z) = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)V \cdot (dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z).$$

$$\Rightarrow -\vec{E} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)V$$

$$-\vec{E} = \nabla V.$$

$$\Rightarrow \boxed{\vec{E} = -\nabla V.}$$

Thus the electric field at any point is given by the negative gradient of potential at that point.

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \quad \text{(Rectangular Co-ordinate)}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \quad \text{(Cylindrical Co-ordinate)}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi \quad \text{(spherical coordinate}$$

③ Given a) $V = E_0 e^{-x} \sin\left(\frac{\pi y}{4}\right)$

b) $V = E_0 r \cos\theta$.

find $\vec{E}$ at point $P(0,1,1)$.

:- a) $V = E_0 e^{-x} \sin\left(\frac{\pi y}{4}\right)$.

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right].$$

$$\frac{\partial V}{\partial x} = E_0 \sin\left(\frac{\pi y}{4}\right)(-1)e^{-x}.$$

$$\frac{\partial V}{\partial y} = E_0 e^{-x} \cos\left(\frac{\pi y}{4}\right)(\pi/4).$$

$$\frac{\partial V}{\partial z} = 0.$$

$$\vec{E} = -\left[-E_0 e^{-x}\sin\left(\frac{\pi y}{4}\right)\hat{a}_x + E_0 e^{-x}\frac{\pi}{4}\cos\left(\frac{\pi y}{4}\right)\hat{a}_y\right] \text{ V/m}.$$

At $P(0,1,1) \Rightarrow \vec{E} = E_0[0.7071\hat{a}_x - 0.555\hat{a}_y]$ V/m.

b). $V = E_0 r\cos\theta$.

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right].$$

$$\frac{\partial V}{\partial r} = E_0\cos\theta, \quad \frac{\partial V}{\partial \theta} = -E_0 r\sin\theta, \quad \frac{\partial V}{\partial \phi} = 0.$$

$$\vec{E} = -E_0\cos\theta\,\hat{a}_r + E_0\sin\theta\,\hat{a}_\theta \text{ V/m}.$$

Convert $P(0,1,1)$ to spherical coordinates.

$r = \sqrt{x^2+y^2+z^2} = \sqrt{2}$.

$\phi = \tan^{-1}(y/x) = \tan^{-1}(1/0) = \pi/2$.

$\theta = \cos^{-1}(z/r) = \cos^{-1}(1/\sqrt{2}) = 45^\circ$.

$$\vec{E} = E_0[-0.7071\hat{a}_r + 0.7071\hat{a}_\phi] \text{ V/m}.$$

(4) An electric potential is given by $V = \dfrac{60\sin\theta}{r^2}$ V find V and $\vec{E}$ at $P(3, 60°, 25°)$.

:- At $P(3, 60°, 25°)$, $r = 3$, $\theta = 60°$, $\phi = 25°$.

$$V = \frac{60\sin 60}{(3)^2} = \underline{\underline{5.7735 \text{ V}}}.$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$$

$$\frac{\partial V}{\partial r} = 60\sin\theta(-2)r^{-3} = \frac{-120\sin\theta}{r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{60}{r^2}\cos\theta \quad , \quad \frac{\partial V}{\partial \phi} = 0 .$$

$$\vec{E} = -\left[-\frac{120\sin\theta}{r^3}\hat{a}_r + \frac{1}{r}\cdot\frac{60}{r^2}\cos\theta\,\hat{a}_\theta\right].$$

At P, $$\vec{E} = -\left[\frac{-120\sin 60°}{3^3}\hat{a}_r + \frac{60}{3^3}\cos 60°\,\hat{a}_\theta\right].$$

$$\boxed{\vec{E} = 3.849\hat{a}_r - 1.11\hat{a}_\theta \text{ V/m}}.$$

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(5) :- If $V = 2x^2y + 20z - \dfrac{4}{x^2+y^2}$. Find $\vec{E}$, $\vec{D}$ and ρ_v at $P(6, -2.5, 3)$.

:- $$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 2y(2x) + 0 - 4\left[\frac{-2x}{(x^2+y^2)^2}\right] = 4xy + \frac{8x}{(x^2+y^2)^2}.$$

$$\frac{\partial V}{\partial y} = 2x^2 + 0 - 4\left[\frac{-2y}{(x^2+y^2)^2}\right] = 2x^2 + \frac{8y}{(x^2+y^2)^2}.$$

$$\frac{\partial V}{\partial z} = 0 + 20 - 0 = 20 .$$

$$\vec{E} = -\left\{\left[4xy + \frac{8x}{(x^2+y^2)^2}\right]\hat{a}_x + \left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right]\hat{a}_y + 20\hat{a}_z\right\}$$

$\vec{E}$ at P $= \{ [-60+0.0268]\hat{a}_x + [72-0.0112]\hat{a}_y + 20\hat{a}_z \}$ (16)

$$\boxed{\vec{E} = 59.9732\hat{a}_x - 71.9888\hat{a}_y - 20\hat{a}_z \ \text{V/m}}$$

$\vec{D}$ at P $=$ $\vec{E}$ at P $\times \epsilon_0$.

$$\boxed{\vec{D} = 0.531\hat{a}_x - 0.6373\hat{a}_y - 0.177\hat{a}_z \ \text{nC/m}^2}$$

$\rho_v = \nabla \cdot \vec{D}$, $\vec{D} = \epsilon_0 \vec{E}$.

$$\rho_v = \epsilon_0 (\nabla \cdot \vec{E})$$

$$= \epsilon_0 \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right].$$

$$= \epsilon_0 \left\{ -\frac{\partial}{\partial x}\left[4xy + \frac{8x}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial y}\left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial z}(20) \right\}$$

$$= \epsilon_0 \left\{ -4y - \frac{(x^2+y^2)^2 8 - 8x \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4} - 0 - \frac{(x^2+y^2)^2(8) - 8y\, 2(x^2+y^2)(2y)}{(x^2+y^2)^4} \right\}$$

$$= \epsilon_0 \left\{ -4y - \frac{8}{(x^2+y^2)^2} + \frac{32x^2}{(x^2+y^2)^3} - \frac{8}{(x^2+y^2)^2} + \frac{32y^2}{(x^2+y^2)^3} \right\}.$$

At P, $x=6$, $y=-25$, $z=3$

$$\boxed{\nabla \cdot \vec{E} = 88.6193 \ \text{pC/m}^3}$$

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(4)